Loops in Reeb Graphs Over 2-Manifolds

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Motivation

- Description of surface embeddings
  [Yoshihisa, Kunii, & Kergosian], 1991
- Fast isocontouring
  [van Kreveld, et. al.], 1997
- Isosurface selection
  [Bajaj, Pascucci, & Schikore], 1997

Previous Work

- O(n^2) algorithm.
  [Shinagawa & Kunii], 1991

- O(n log n) algorithm. Approximate.
  [Hilaga, et. al.], 2001

- Contour tree in O(n log n + t □(t)).
  [Carr, Snoeyink, & Axen], 2001

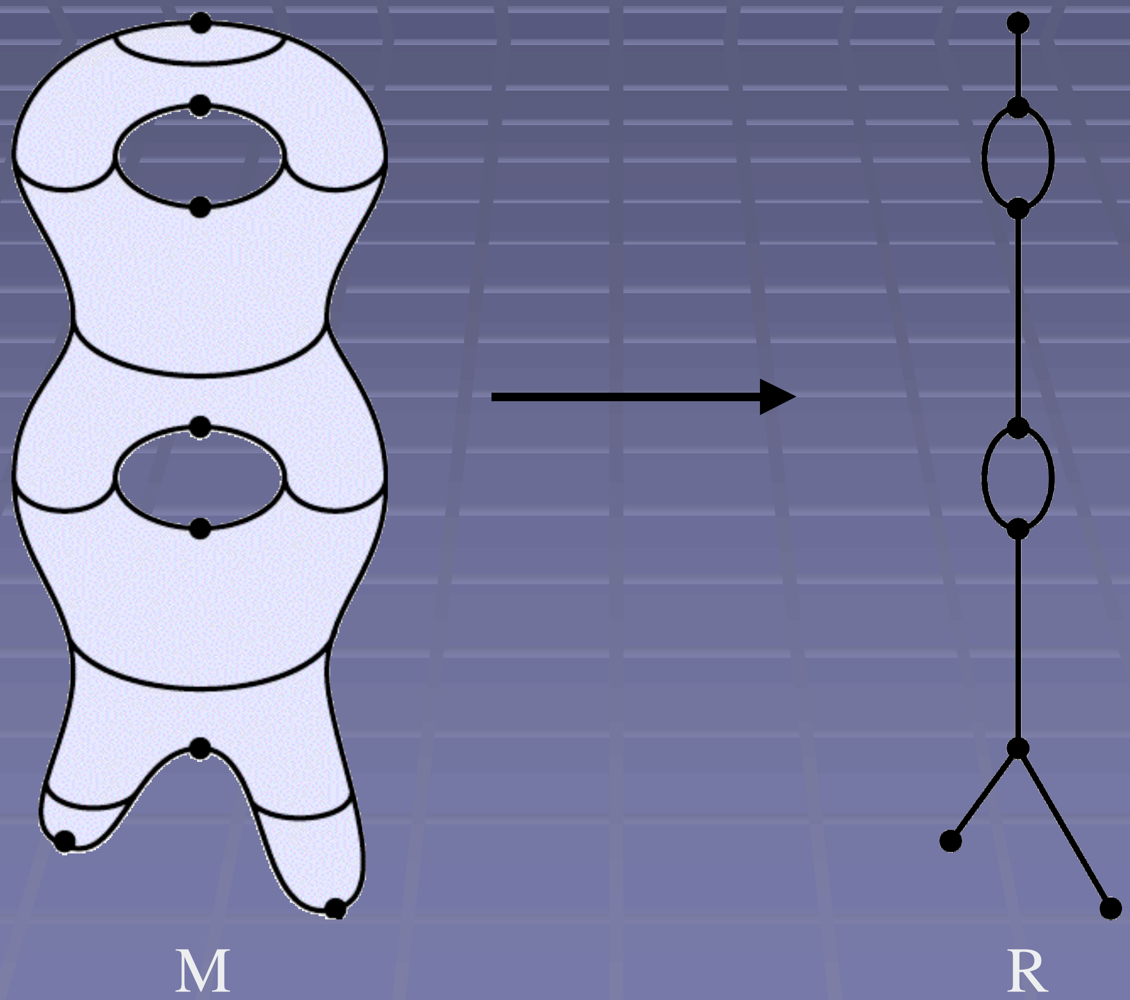
- Augmented Contour tree in O(n + t □(t)). Parallel algorithm.
  [Cole-McLaughlin & Pascucci], 2002

Definition

- A **Morse function**, f, over a manifold, M, is a real-valued function with distinct critical values.

- The **Reeb Graph**, R, of f is obtained by contracting the components of $f^{-1}(x)$ to points.

Genus 2 Surface



Characterization of Topology

- Orientable surfaces are characterized by genus, g, and number of boundary components, h.

- M = S # T # ••• # T g tori.

- Euler characteristic, $\chi = 2 - 2g + h$

Characterization of Topology

- Number of index k critical points of $f = c_k$

  $\chi = c_0 - c_1 + c_2$

- The k'th Betti number of M = rank $H_k(M) = \beta_k$

  $\chi = \beta_0 - \beta_1 + \beta_2$

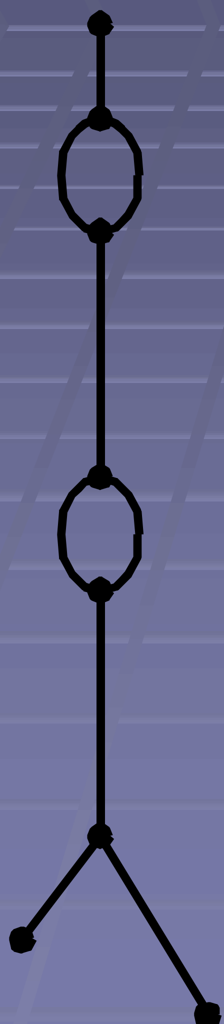
- Surjection from $H_1(M)$ onto $H_1(R)$.

- # of loops = rank $H_1(R) \leq$ rank $H_1(M) = \beta_1$

Closed, Orientable Surafces

- # of loops in $R = g$.
- Construct R' from R with the same homotopy type.
  - Remove degree 1 node (index 0 or 2).
  - Remove degree 2 node (index 1).
- Count remaining degree 3 nodes.

Closed, Orientable Surfaces



- n_3 = # of degree 3 nodes in *R'*.

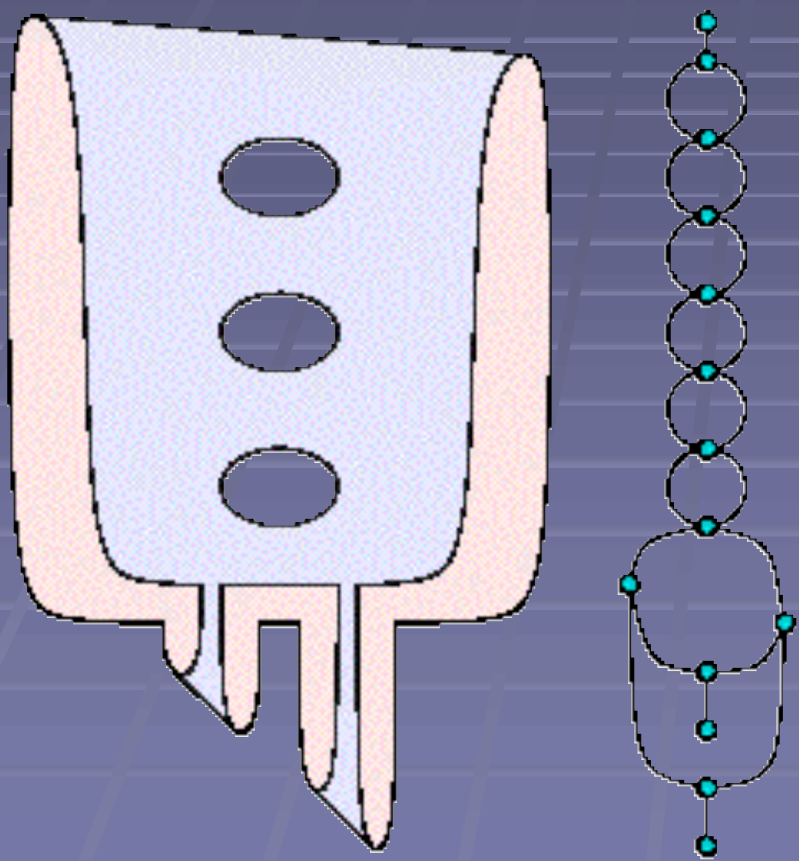
- $n_3 = c_1 - (c_0 + c_2) = -\chi_M = 2g - 2$

- # loops of *R'* = $(n_3 / 2) + 1 = g$.

Open, Orientable Surfaces

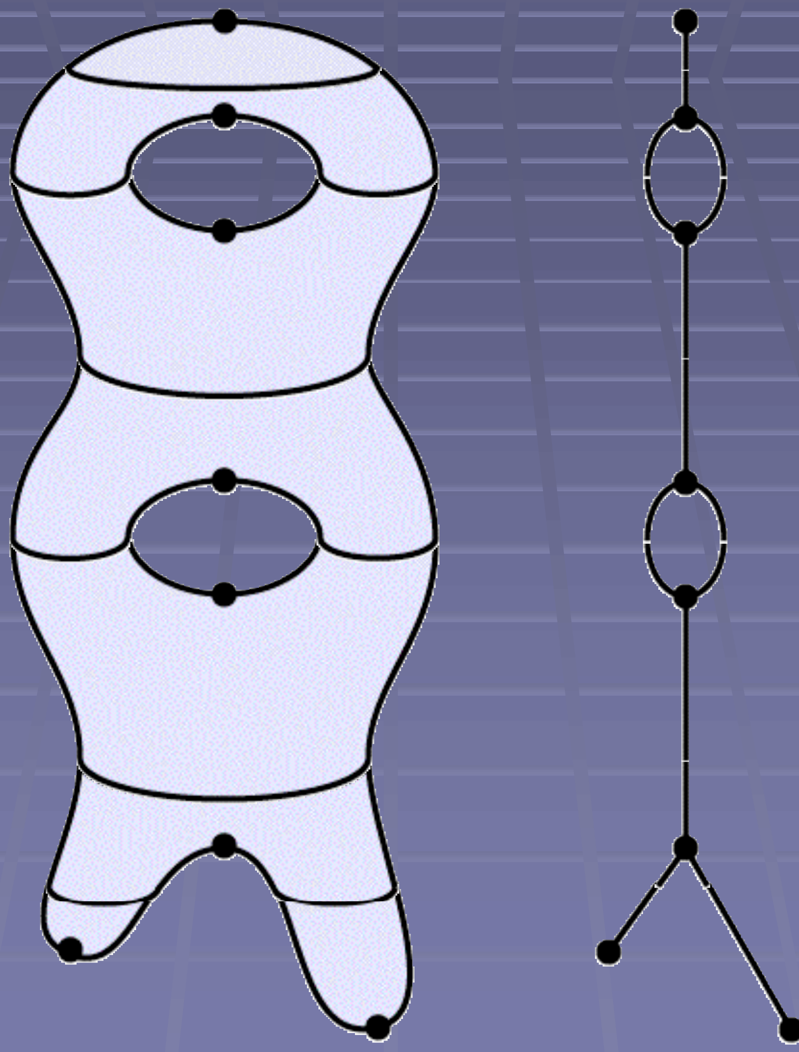
- Construct a closed surface by gluing *h* discs to each boundary component.
- Gluing the discs either removes loops from *R* or not.
- g < # of loops < $2g + h - 1 = \square_1$

Open, Orientable Surfaces



- Lower bound: No two boundary components intersect the same contour cycle.
- Upper bound: Example to the left realizes the upper bound.
- Every boundary component creates a loop in the Reeb graph.

Reeb Graph Algorithm



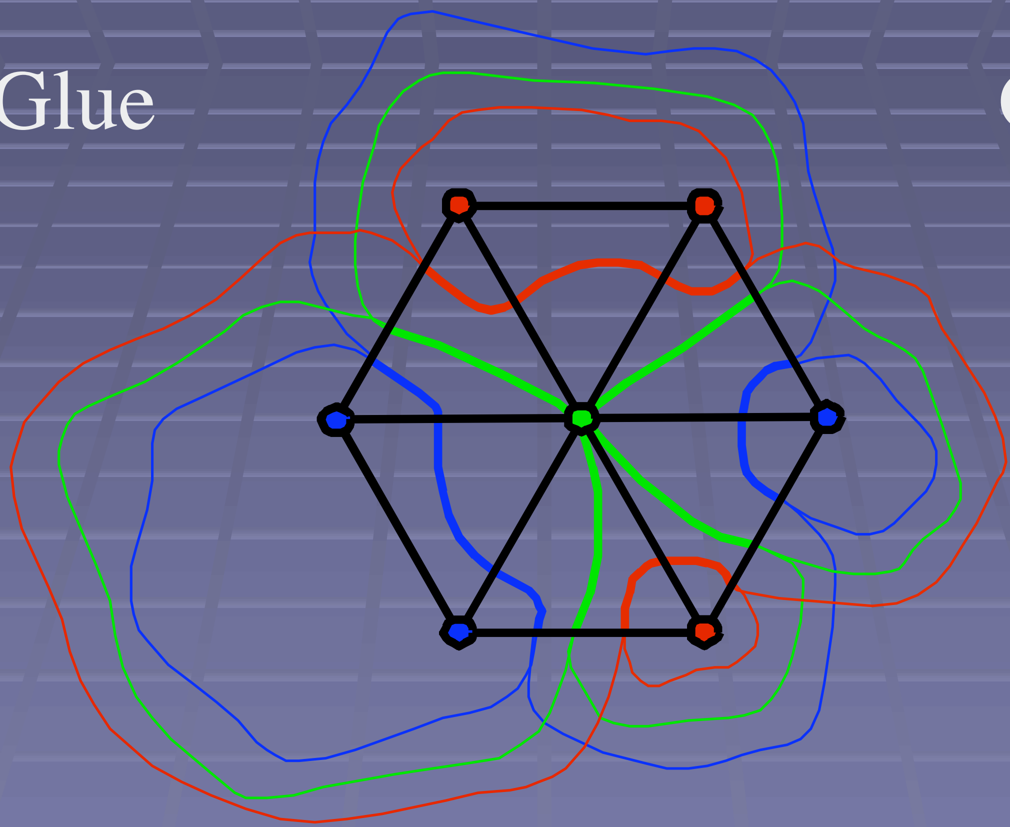
Reeb Graph Algorithm

- Data Structures: Cycles, Paths.
- Create/Destroy Cycles at extrema.
- Cut/Glue Cycles/Paths at saddle points.
- Problem: How to count Cycles/Paths?

Cut/Glue Operations

Glue

Cut



Isovalues:

0	1	2

Reeb Graph Algorithm

- Solution: Store Cycles/Paths in an efficient data structure.
- Inefficient operation: *Find*.
- Balanced search tree or randomized search tree.
  - *Find* has complexity O(log n).

Reeb Graph Algorithm

- Store edges of Cycles/Paths at the nodes of the search tree.
- At each saddle point use *Find* to decide which edges of *R* to merge or split.
- Complexity O(n log n) to construct Reeb Graph.

Future Work

- Is this algorithm optimal?
- Reeb graphs over 3-manifolds.

Reeb Graph Algorithm

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