Loops in Reeb Graphs Over 2-Manifolds

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Motivation

- Description of surface embeddings
  [Yoshihisa, Kunii, & Kergosian], 1991

- Fast isocontouring
  [van Kreveld, et. al.], 1997

- Isosurface selection
  [Bajaj, Pascucci, & Schikore], 1997
Previous Work

- $O(n^2)$ algorithm.
  [Shinagawa & Kunii], 1991

- $O(n \log n)$ algorithm. Approximate.
  [Hilaga, et. al.], 2001

- Contour tree in $O(n \log n + t \, o(t))$.
  [Carr, Snoeyink, & Axen], 2001

- Augmented Contour tree in $O(n + t \, o(t))$.
  Parallel algorithm.
  [Cole-McLaughlin & Pascucci], 2002
Definition

- A Morse function, $f$, over a manifold, $M$, is a real-valued function with distinct critical values.

- The Reeb Graph, $R$, of $f$ is obtained by contracting the components of $f^{-1}(x)$ to points.
Genus 2 Surface

M

R
Characterization of Topology

- Orientable surfaces are characterized by genus, $g$, and number of boundary components, $h$.

- $M = S \# T \# \cdots \# T \ g \text{ torii.}$

- Euler characteristic, $\chi = 2 - 2g + h$
Characterization of Topology

- Number of index k critical points of \( f = c_k \)
  \[ \# = c_0 - c_1 + c_2 \]

- The k’th Betti number of \( M = \text{rank } H_k(M) = \#_k \)
  \[ \# = \#_0 - \#_1 + \#_2 \]

- Surjection from \( H_1(M) \) onto \( H_1(R) \).

- # of loops = \( \text{rank } H_1(R) \leq \text{rank } H_1(M) = \#_1 \)
Closed, Orientable Surfaces

- # of loops in $R = g$.

- Construct $R'$ from $R$ with the same homotopy type.
  - Remove degree 1 node (index 0 or 2).
  - Remove degree 2 node (index 1).

- Count remaining degree 3 nodes.
Closed, Orientable Surfaces

- \( n_3 = \# \text{ of degree 3 nodes in } R' \).

- \( n_3 = c_1 - (c_0 + c_2) = -\chi_M = 2g - 2 \)

- \# loops of \( R' = (n_3 / 2) + 1 = g \).
Open, Orientable Surfaces

- Construct a closed surface by gluing $h$ discs to each boundary component.

- Gluing the discs either removes loops from $R$ or not.

- $g < \# \text{ of loops} < 2g + h - 1 = \mu_1$
Open, Orientable Surfaces

- Lower bound: No two boundary components intersect the same contour cycle.
- Upper bound: Example to the left realizes the upper bound.
- Every boundary component creates a loop in the Reeb graph.
Reeb Graph Algorithm
Reeb Graph Algorithm

- Data Structures: Cycles, Paths.
- Create/Destroy Cycles at extrema.
- Cut/Glue Cycles/Paths at saddle points.
- Problem: How to count Cycles/Paths?
Cut/Glue Operations
Reeb Graph Algorithm

- Solution: Store Cycles/Paths in an efficient data structure.

- Inefficient operation: \textit{Find}.

- Balanced search tree or randomized search tree.
  - \textit{Find} has complexity $O(\log n)$. 
Reeb Graph Algorithm

- Store edges of Cycles/Paths at the nodes of the search tree.

- At each saddle point use \textit{Find} to decide which edges of $R$ to merge or split.

- Complexity $O(n \log n)$ to construct Reeb Graph.
Future Work

- Is this algorithm optimal?
- Reeb graphs over 3-manifolds.
Reeb Graph Algorithm

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